

A NONEXTENSION RESULT ON THE SPECTRAL METRIC

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ABSTRACT. The spectral metric, defined by Schwarz and Oh using Floer-theoretical method, is a bi-invariant metric on the Hamiltonian diffeomorphism group. We show in this note that for certain symplectic manifolds, this metric can not be extended to a bi-invariant metric on the full group of symplectomorphisms. We also study the bounded isometry conjecture of Lalonde and Polterovich in the context of the spectral metric. In particular, we show that the conjecture holds for the torus with all linear symplectic forms.

1. INTRODUCTION

For a closed symplectic manifold (M, ω) , the group $\text{Symp}(M, \omega)$ of all symplectomorphisms has two important subgroups, namely, its identity component $\text{Symp}_0(M, \omega)$ and the subgroup $\text{Ham}(M, \omega)$ of all Hamiltonian diffeomorphisms. These two subgroups coincide when the first cohomology group $H^1(M, \mathbb{R})$ of M vanishes, and they behave quite differently when $H^1(M, \mathbb{R})$ does not vanish. For instance, a Hamiltonian diffeomorphism always has a fixed point according to the Arnold conjecture, but a symplectomorphism does not necessarily have one.

Another example is that the group $\text{Ham}(M, \omega)$ admits several interesting bi-invariant metrics, namely, the famous Hofer metric discovered by Hofer [8] and the spectral metric defined by Schwarz [18] and Oh [15] using Floer-theoretical method, which we will describe in Section 2. For the group $\text{Symp}_0(M, \omega)$, although it also admits bi-invariant metrics (cf. Prop 1.2.A [9] and Prop 3.10 [7]), these metrics are not quite naturally defined.

One way to construct interesting metrics on $\text{Symp}_0(M, \omega)$ is to extend the existing metrics such as the Hofer metric and the spectral metric on $\text{Ham}(M, \omega)$. In fact, such extensions are constructed in [2, 3] for certain symplectic manifolds. While the metrics on $\text{Ham}(M, \omega)$ are bi-invariant, they only extend to right-invariant metrics on $\text{Symp}_0(M, \omega)$. In the opposite direction, it is proven in [7] that the Hofer metric does not extend bi-invariantly to $\text{Symp}_0(M, \omega)$ for certain symplectic manifolds. In fact, it is conjectured that this is true in general.

In this note, we focus on the spectral metric. In particular, using ideas of Lalonde-Polterovich [9] and Schwarz [18], we prove the following result. A submanifold $L \hookrightarrow M$ is incompressible if the inclusion induces an injection on the fundamental groups.

Theorem 1.1. *Let (M, ω) be closed and symplectically aspherical manifold, and $L \hookrightarrow (M, \omega)$ be an incompressible Lagrangian torus. If there exists some $\phi \in$*

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$\text{Symp}_0(M, \omega)$ such that $\phi(L) \cap L = \emptyset$, then the spectral metric does not extend to a bi-invariant metric on $\text{Symp}_0(M, \omega)$.

Remark 1.2. The assumption that M is symplectically aspherical can be removed, as the spectral metric is defined for general closed symplectic manifolds. However, we decide to keep it here since we only describe the definition of the spectral metric for symplectically aspherical manifolds in this paper.

The following corollary is immediate. Here $(\mathbb{T}^{2n}, \omega_0)$ stands for the torus with the standard form.

Corollary 1.3. *The spectral metric on $\text{Ham}(\mathbb{T}^{2n}, \omega_0)$ does not extend to a bi-invariant metric on $\text{Symp}_0(\mathbb{T}^{2n}, \omega_0)$.*

Note that Theorem 1.1 does not directly apply to the torus $(\mathbb{T}^{2n}, \omega_L)$ with arbitrary linear symplectic forms ω_L since there may not even exist such closed Lagrangian submanifolds at all. The following result is proved using a similar but somewhat different method.

Theorem 1.4. *The spectral metric on $\text{Ham}(\mathbb{T}^{2n}, \omega_L)$ does not extend to a bi-invariant metric on $\text{Symp}_0(\mathbb{T}^{2n}, \omega_L)$ for the torus \mathbb{T}^{2n} with any linear symplectic form ω_L .*

One may conjecture as in the case for the Hofer norm that the above theorems hold for general closed symplectic manifolds, although all indications are that this problem is out of reach using currently known methods.

Conjecture 1.5. *For any closed symplectic manifold (M, ω) such that $\text{Symp}_0(M, \omega)$ is not identical to $\text{Ham}(M, \omega)$, the spectral metric on $\text{Ham}(M, \omega)$ does not extend to a bi-invariant metric on $\text{Symp}_0(M, \omega)$.*

A related question is the bounded isometry conjecture proposed by Lalonde and Polterovich in [9]. A symplectomorphism $\phi \in \text{Symp}_0(M, \omega)$ is said to be bounded with respect to the Hofer norm $\|\cdot\|$ if

$$r_{\|\cdot\|}(\phi) := \sup_{f \in \text{Ham}(M, \omega)} \{\|f\phi f^{-1}\phi^{-1}\|\} < \infty.$$

Let $\text{BI}_0(M, \omega, \|\cdot\|)$ be the set of all bounded symplectomorphisms with respect to the Hofer norm in $\text{Symp}_0(M, \omega)$. The bounded isometry conjecture for the Hofer norm states that

$$\text{BI}_0(M, \omega, \|\cdot\|) = \text{Ham}(M, \omega) \text{ for all } (M, \omega).$$

As pointed out in [7], this is a very hard question in general. Some partial results can be found in [6, 9, 10]. Now the bounded isometry conjecture can easily be formulated with respect to the spectral norm γ .

Conjecture 1.6. $\text{BI}_0(M, \omega, \gamma) = \text{Ham}(M, \omega)$ for all (M, ω) .

Both conjectures are trivial if $H^1(M, \mathbb{R})$ vanishes. When $H^1(M, \mathbb{R})$ does not vanish, a necessary condition for both conjectures to hold is that $\text{Ham}(M, \omega)$ has infinite diameter with respect to the spectral norm γ . This diameter is called the spectral capacity of M (cf. Equation (2.47) in Albers [1]). As noticed by Entov-Polterovich [4] and explicitly pointed out by McDuff [12], the spectral capacity of certain manifolds such as $\mathbb{C}P^n$ is finite. On the other hand, the spectral capacity of \mathbb{T}^{2n} is infinite. As far as I know, there are no known examples of closed symplectic

manifolds with finite spectral capacity and such that $H^1(M, \mathbb{R})$ does not vanish. Such examples would immediately disprove both conjectures.

As one will see below, Conjecture 1.6 implies Conjecture 1.5. Note also that Conjecture 1.6 implies the corresponding conjecture for the Hofer norm, as the Hofer norm is greater than the spectral norm. Therefore, it seems impossible to prove them in general at this time. In this note, we only prove it for the torus with linear symplectic forms.

Theorem 1.7. *The bounded isometry conjecture for the spectral norm holds for the torus \mathbb{T}^{2n} with all linear symplectic forms ω_L .*

Organization of the paper. In Section 2 we review the definition of the spectral metric and its properties. We prove Theorem 1.1 in Section 3 using an explicit construction similar to that of Schwarz [18]. Then we prove Theorem 1.7 in Section 4. Theorem 1.4 is a corollary of Theorem 1.7.

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2. THE SPECTRAL METRIC

In this section we review without proofs the necessary ingredients to define the spectral metric. The details can be found in Schwarz [18] for the case of closed symplectically aspherical manifolds, and in Oh [14, 15] for closed symplectic manifolds where ω is rational. See also Usher [19] in the case ω is not necessarily rational. For our purpose and for simplicity, we only consider closed symplectically aspherical manifolds, and will mainly follow Schwarz [18] for the definition. A symplectic manifold (M, ω) is symplectically aspherical if

$$c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} = 0.$$

2.1. Floer homology. Let (M, ω) be a closed symplectically aspherical manifold. Let $H_t = H_{t+1} : M \rightarrow \mathbb{R}$ be a time-dependent 1-periodic Hamiltonian function. The time-dependent Hamiltonian vector field X_{H_t} is given by $\iota(X_{H_t}) = dH_t$. Consider the Hamiltonian differential equation

$$(2.1) \quad \dot{x}(t) = X_{H_t}(x(t)).$$

The solution of (2.1) generates a family of Hamiltonian diffeomorphisms $\phi_H^t : M \rightarrow M$ via

$$\frac{d}{dt}\phi_H^t = X_{H_t} \circ \phi_H^t \text{ and } \phi_H^0 = \text{id}.$$

We shall first assume H is regular, i.e. all fixed points of the time-1 map ϕ_H^1 are non-degenerate.

Given a time-dependent 1-periodic Hamiltonian function H , the action functional \mathcal{A}_H on the space \mathcal{LM} of contractible loops in M is given by

$$\mathcal{A}_H(x) = \int_B u^* \omega + \int_0^1 H_t(x(t)) dt$$

for $x \in \mathcal{LM}$, where B is the unit disk and $u : B \rightarrow M$ is any extension of x . For symplectically aspherical manifold (M, ω) , the action functional \mathcal{A}_H takes values in \mathbb{R} . The critical points of \mathcal{A}_H are contractible 1-periodic solutions of (2.1), i.e.

$$\text{Crit}\mathcal{A}_H = \mathcal{P}(H) := \{x \in \mathcal{LM} \mid x \text{ satisfies (2.1)}\}.$$

The assumption that H is regular implies that $\mathcal{P}(H)$ is a finite set. The Conley-Zehnder index $\mu_{\text{CZ}} : \mathcal{P}(H) \rightarrow \mathbb{Z}$ takes values in \mathbb{Z} , and it can be normalized such that

$$\mu_{\text{CZ}}(x) = \mu_{\text{Morse}}(x)$$

for C^2 -small time-independent Morse function H , where $x \in \mathcal{P}(H) = \text{Crit}H$.

Let J be an ω -compatible almost complex structure on M . We consider the moduli spaces $\mathcal{M}_{x,y}(J, H)$ of Floer trajectories connecting $x, y \in \mathcal{P}(H)$. Namely,

$$\begin{aligned} \mathcal{M}_{x,y}(J, H) := \{u : \mathbb{R} \times S^1 \rightarrow M \mid \partial_s u + J(u)(\partial_t u - X_{H_t}(u)) = 0, \\ \lim_{s \rightarrow -\infty} u(s, t) = x, \lim_{s \rightarrow +\infty} u(s, t) = y\}. \end{aligned}$$

Each solution $u \in \mathcal{M}_{x,y}(J, H)$ has finite energy

$$E(u) := \int_{-\infty}^{\infty} \int_0^1 |\partial_s u|^2 ds dt = \mathcal{A}_H(y) - \mathcal{A}_H(x) \geq 0.$$

For generic (J, H) , $\mathcal{M}_{x,y}(J, H)$ is a smooth manifold of dimension

$$\dim \mathcal{M}_{x,y}(J, H) = \mu_{\text{CZ}}(y) - \mu_{\text{CZ}}(x).$$

When $\mu_{\text{CZ}}(y) - \mu_{\text{CZ}}(x) = 1$, the space $\mathcal{M}_{x,y}(J, H)$ is a 1-dimensional manifold, and the quotient $\mathcal{M}_{x,y}(J, H)/\mathbb{R}$ is a set of finitely many isolated points. Here the \mathbb{R} -action is the shift in s -variable. We denote $n(x, y) := \#\mathcal{M}_{x,y}(J, H)/\mathbb{R}$, where the connecting orbits are counted with appropriate signs.

Now consider the Floer chain complex $CF_*(H) = \mathcal{P}(H) \otimes \mathbb{Z}$ with the grading given by the conley-Zehnder index μ_{CZ} . The boundary operator is defined as

$$\partial : CF_*(H) \rightarrow CF_*(H),$$

$$\partial y = \sum_{\mu_{\text{CZ}}(x) = \mu_{\text{CZ}}(y) - 1} n(x, y) x.$$

By definition, the boundary operator ∂ has degree -1 .

Floer [5] proved that $\partial \circ \partial = 0$. Its homology groups are called Floer homology groups of the pair (J, H) , and are denoted by $HF_*(J, H)$. Floer also proved that $HF_*(J, H)$ are independent of (J, H) , and are canonically isomorphic to the singular homology groups of M .

2.2. The PSS isomorphism. Let $f : M \rightarrow \mathbb{R}$ be a Morse function, and g be a generic Riemannian metric. Let $\rho : \mathbb{R} \rightarrow [0, 1]$ be any smooth cut-off function such that $\rho(s) = 1$ for $s \leq -1$, and $\rho(s) = 0$ for $s \geq 1$. For $x \in \mathcal{P}(H)$ and $y \in \text{Crit}f$, define the moduli spaces

$$\begin{aligned} \mathcal{M}_{x,y}^+(J, H; f, g) := \{(u, \gamma) \mid u : \mathbb{R} \times S^1 \rightarrow M, \gamma : [0, \infty) \rightarrow M, \\ \partial_s u + J(u)(\partial_t u - \rho(s)X_{H_t}(u)) = 0, \dot{\gamma} + \nabla_g f \circ \gamma = 0, \\ \int_{-\infty}^{\infty} \int_0^1 |\partial_s u|^2 ds dt < \infty, \\ \lim_{s \rightarrow -\infty} u(s, t) = x, \lim_{s \rightarrow \infty} u(s, t) = \gamma(0), \lim_{s \rightarrow \infty} \gamma(s) = y\}. \end{aligned}$$

Similarly, define

$$\begin{aligned}\mathcal{M}_{y,x}^-(J, H; f, g) &:= \{(\gamma, u) \mid \gamma : (-\infty, 0] \rightarrow M, u : \mathbb{R} \times S^1 \rightarrow M, \\ &\quad \dot{\gamma} + \nabla_g f \circ \gamma = 0, \partial_s u + J(u)(\partial_t u - \rho(-s)X_{H_t}(u)) = 0, \\ &\quad \int_{-\infty}^{\infty} \int_0^1 |\partial_s u|^2 ds dt < \infty, \\ &\quad \lim_{s \rightarrow -\infty} \gamma(s) = y, \gamma(0) = \lim_{s \rightarrow -\infty} u(s, t), \lim_{s \rightarrow \infty} u(s, t) = x\}.\end{aligned}$$

These spaces are often called moduli spaces of J -holomorphic spiked discs, where the spike is the gradient flow line between $y \in \text{Crit} f$ and $u(0)$. For generic choices of (J, H, f, g) , the space $\mathcal{M}_{x,y}^+(J, H; f, g)$ is a manifold of dimension

$$\dim \mathcal{M}_{x,y}^+(J, H; f, g) = \mu_{\text{Morse}}(y) - \mu_{\text{CZ}}(x),$$

and the space $\mathcal{M}_{y,x}^-(J, H; f, g)$ is a manifold of dimension

$$\dim \mathcal{M}_{y,x}^-(J, H; f, g) = \mu_{\text{CZ}}(x) - \mu_{\text{Morse}}(y).$$

Moreover, in dimension 0 they are compact, hence finite.

Define Φ_{MF} from $CM_*(f)$ to $CF_*(H)$ by

$$\Phi_{\text{MF}}(y) = \sum_{\mu_{\text{CZ}}(x) = \mu_{\text{Morse}}(y)} \# \mathcal{M}_{x,y}^+(J, H; f, g)x,$$

and Φ_{FM} from $CF_*(H)$ to $CM_*(f)$ by

$$\Phi_{\text{FM}}(x) = \sum_{\mu_{\text{Morse}}(y) = \mu_{\text{CZ}}(x)} \# \mathcal{M}_{y,x}^-(J, H; f, g)y.$$

Here the counting is again with appropriate signs. These maps commute with boundary operators, hence they induce homomorphisms

$$\Phi_{\text{MF}} : HM_*(f, g) \rightarrow HF_*(J, H)$$

and

$$\Phi_{\text{FM}} : HF_*(J, H) \rightarrow HM_*(f, g).$$

In fact they are isomorphisms, known as the PSS isomorphisms (cf. [17, 11] for instance), and

$$\Phi_{\text{MF}} = \Phi_{\text{FM}}^{-1}.$$

2.3. The spectral metric. Given any $a \in \mathbb{R}$, define

$$\mathcal{P}^a(H) = \{x \in \mathcal{P}(H) \mid \mathcal{A}_H(x) \leq a\}$$

and

$$C_*^a(H) = \mathcal{P}^a(H) \otimes \mathbb{Z}.$$

Note that $C_*^a(H)$ is invariant under the Floer boundary operator ∂ . This means $(C_*^a(H), \partial)$ is a sub-complex of $(C_*(H), \partial)$, which is called filtered Floer chain complex. The induced homology groups $HF_*^a(J, H)$ are called filtered Floer homology groups.

The obvious inclusion map

$$\iota_*^a : C_*^a(H) \rightarrow C_*(H)$$

induces

$$\iota_*^a : HF_*^a(J, H) \rightarrow HF_*(J, H).$$

It is clear that for a sufficiently large, $HF_*^a(J, H) = HF_*(J, H)$ and ι_*^a is the identity map; for a sufficiently small, $HF_*^a(J, H) = 0$ and $\iota_*^a = 0$.

Now given any nonzero homology class $\alpha \in H_*(M)$, under the isomorphism $H_*(M) \cong HM_*(f, g)$, we can think of α as an element in $HM_*(f, g)$. Define

$$C_\alpha(H) = \inf \{a \in \mathbb{R} \mid \Phi_{\text{MF}}(\alpha) \in \text{im } \iota_*^a\}.$$

Note that $C_\alpha(H)$ are finite numbers, and are critical values of the action functional \mathcal{A}_H . Note also that they only depends on H , but not on (f, g) .

A standard energy estimate (cf. [18]) shows that

$$(2.2) \quad E_-(H) \leq C_\alpha(H) \leq E_+(H)$$

where

$$E_-(H) = \int_0^1 \min_{x \in M} H_t(x) dt, \quad \text{and} \quad E_+(H) = \int_0^1 \max_{x \in M} H_t(x) dt.$$

A similar energy estimate shows that C_α satisfies

$$C_\alpha(H) - C_\alpha(K) \leq E_+(H - K).$$

This inequality implies C_α is continuous with respect to the semi-norm

$$\|H\| = E_+(H) - E_-(H),$$

and can therefore be extended continuously to non-regular Hamiltonians (Prop 2.14 in [18]). Thus we do not assume the regularity of H any more from now on.

A deep result is that for a symplectically aspherical manifold (M, ω) ,

$$C_\alpha(H) = C_\alpha(K) \quad \text{whenever} \quad \phi_H^1 = \phi_K^1$$

where ϕ_H^1 and ϕ_K^1 are time-1 maps of the Hamiltonians H and K , respectively. Thus C_α descends to $\text{Ham}(M, \omega)$ by defining

$$C_\alpha(\phi) = C_\alpha(H) \quad \text{if} \quad \phi_H^1 = \phi.$$

Note that in general, C_α only descends to the universal cover $\widetilde{\text{Ham}}(M, \omega)$ (See [14, 15, 19] for details). In both cases, C_α are called the spectral invariants.

Definition 2.1. *The spectral norm $\gamma : \text{Ham}(M, \omega) \rightarrow \mathbb{R}$ is defined as*

$$\gamma(\phi) = C_{[M]}(\phi) - C_{[pt]}(\phi)$$

where $[M] \in H_{2n}(M)$ and $[pt] \in H_0(M)$ are the respective generators. The induced metric

$$d_\gamma(\phi, \psi) = \gamma(\phi\psi^{-1})$$

is called the spectral metric.

Remark 2.2. Since we are using homology $H_*(M)$ rather than cohomology $H^*(M)$, our definition is slightly different from, but equivalent to that of [18]. We refer the reader to [14, 15, 19] for the definition of the spectral metric for general closed symplectic manifolds.

The following properties of the spectral norm γ follow directly from the properties of the spectral invariants C_α .

- (Nondegeneracy) $\gamma(\phi) \geq 0$, and $\gamma(\phi) = 0 \Rightarrow \phi = id$.
- (Symmetry) $\gamma(\phi) = \gamma(\phi^{-1})$.
- (Triangle Inequality) $\gamma(\phi\psi) \leq \gamma(\phi) + \gamma(\psi)$.
- (Conjugate Invariancy) $\gamma(\psi\phi\psi^{-1}) = \gamma(\phi)$.

These properties of γ are equivalent to saying that the spectral metric d_γ is a nondegenerate bi-invariant metric. In particular, γ is conjugate-invariant implies d_γ is bi-invariant. Note that in view of (2.2), the spectral norm γ is bounded from above by the Hofer norm $\|\cdot\|$. That is,

$$\gamma(\phi) \leq \|\phi\|.$$

Here the Hofer norm is another conjugate-invariant norm on $\text{Ham}(M, \omega)$ first discovered by Hofer in [8]. It is defined by

$$\|\phi\| = \inf_{\phi_H^1 = \phi} \left\{ \int_0^1 \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) dt \right\}.$$

The induced metric

$$d_H(\phi, \psi) = \|\phi\psi^{-1}\|$$

is called the Hofer metric which is also bi-invariant.

3. PROOF OF THEOREM 1.1

Recall from [7] that to prove the spectral metric (in fact any bi-invariant metric) d_γ on $\text{Ham}(M, \omega)$ does not extend to a bi-invariant metric on $\text{Symp}_0(M, \omega)$, it suffices to show that there exists some unbounded symplectomorphism $\phi \in \text{Symp}_0(M, \omega)$ with respect to the spectral norm γ . That is, ϕ satisfies

$$r_\gamma(\phi) := \sup_{f \in \text{Ham}(M, \omega)} \{\gamma(f\phi f^{-1}\phi^{-1})\} = \infty.$$

The reason is that assume d_γ extend bi-invariantly, then we have

$$\gamma(f\phi f^{-1}\phi^{-1}) \leq \gamma(f\phi f^{-1}) + \gamma(\phi^{-1}) = \gamma(\phi) + \gamma(\phi^{-1}) = 2\gamma(\phi).$$

This would imply $\gamma(\phi) \geq 2r_\gamma(\phi) = \infty$, which is a contradiction.

Proof of Theorem 1.1. By assumption, there exists some $\phi \in \text{Symp}_0(M, \omega)$ such that $\phi(L) \cap L = \emptyset$. It suffices to show that ϕ is unbounded with respect to the spectral norm γ . To do this, one needs to find a family of Hamiltonian diffeomorphisms f_c such that $\gamma(f_c\phi f_c^{-1}\phi^{-1}) \rightarrow \infty$ as $c \rightarrow \infty$. We follow Schwarz (Example 5.7 in [18]) for the construction of such f_c 's.

The Lagrangian neighborhood theorem (cf. Weinstein [20]) asserts that there exists a neighbourhood $V \subset M$ of L which is symplectomorphic to a neighbourhood $\mathcal{N}(L) \subset T^*L$ of the zero section. Now L is a torus of dimension n , where $2n$ is the dimension of M , so T^*L is symplectomorphic to $(\mathbb{T}^n \times \mathbb{R}^n, \omega_0 = \sum_{i=1}^n dt_i \wedge dr_i)$, where t_i 's and r_i 's are coordinates of \mathbb{T}^n and \mathbb{R}^n respectively. By shrinking V if necessary, we can assume that $\phi(V) \cap V = \emptyset$, and there exists a diffeomorphism $\Phi : V \rightarrow \mathbb{T}^n \times B_\epsilon^n$ such that $\Phi^*\omega_0 = \omega$. Here $B_\epsilon^n \subset \mathbb{R}^n$ is a ball of radius ϵ .

Given any constant c , let $\rho_c : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function with $\text{supp}(\rho_c) \subset B_\epsilon^n$ and its only critical values are 0 and its maximum $\rho(\mathbf{0}) = c > 0$. Define

$$\bar{\rho}_c : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } \bar{\rho}_c(\mathbf{t}, \mathbf{r}) = \rho_c(\mathbf{r}).$$

Using the diffeomorphism $\Phi : V \rightarrow \mathbb{T}^n \times B_\epsilon^n$, we construct a family of time-independent Hamiltonians $F_c : M \rightarrow \mathbb{R}$ such that

$$F_c(x) := \begin{cases} \bar{\rho}_c \circ \Phi(x), & \text{if } x \in V, \\ 0, & \text{otherwise.} \end{cases}$$

Let $f_c \in \text{Ham}(M, \omega)$ be the time-1 map of the Hamiltonian F_c . Since f_c is supported in V and $\phi(V) \cap V = \emptyset$, it is easy to see that

$$h_c := f_c \phi f_c^{-1} \phi^{-1} \in \text{Ham}(M, \omega)$$

is the time-1 map of the Hamiltonian $H_c : M \rightarrow \mathbb{R}$ where

$$H_c(x) := \begin{cases} \bar{\rho}_c \circ \Phi(x), & \text{if } x \in V, \\ -\bar{\rho}_c \circ \Phi \circ \phi^{-1}(x), & \text{if } x \in \phi(V), \\ 0, & \text{otherwise.} \end{cases}$$

Note that since the Lagrangian torus L is incompressible, the only contractible 1-periodic solutions of (2.1) are the critical points of H_c . Thus the action functional \mathcal{A}_{H_c} has exactly 3 distinct critical values $c, 0, -c$. Recall that the spectral norm

$$\gamma(h_c) = C_{[M]}(h_c) - C_{[1]}(h_c) = C_{[M]}(H_c) - C_{[1]}(H_c)$$

where $C_{[M]}(H_c)$ and $C_{[1]}(H_c)$ are two distinct critical values of \mathcal{A}_{H_c} . So we get $\gamma(h_c) \geq c$. Thus $\gamma(h_c) \rightarrow \infty$ as $c \rightarrow \infty$. This proves ϕ is unbounded with respect to γ , hence Theorem 1.1. \square

4. PROOF OF THEOREM 1.4 AND 1.7

In this section, we prove the bounded isometry conjecture with respect to the spectral norm for the torus \mathbb{T}^{2n} with all linear forms ω_L , namely, Theorem 1.7. Theorem 1.4 then follows immediately as a corollary.

We begin with the flux homomorphism $\text{Flux} : \text{Symp}_0(M, \omega) \rightarrow H^1(M, \mathbb{R})$ which is defined by

$$\text{Flux}(\{\phi_t\}) := \int_0^1 \iota(X_t) \omega dt$$

where the vector field X_t is determined by $\frac{d}{dt} \phi_t = X_t \circ \phi_t$. It induces

$$\text{Flux} : \text{Symp}_0(M, \omega) \rightarrow H^1(M, \mathbb{R}) / \Gamma_\omega$$

where $\Gamma_\omega := \text{Flux}(\pi_1(\text{Symp}_0(M, \omega))) \subset H^1(M, \mathbb{R})$ is called the flux subgroup, which is always discrete by Ono [16]. It is well known that the map Flux is surjective, and its kernel is equal to $\text{Ham}(M, \omega)$. Thus two symplectomorphisms have the same flux if and only if they differ by a Hamiltonian diffeomorphism. See [13] Chapter 10 for more details.

Recall that $\phi \in \text{Symp}_0(M, \omega)$ is bounded with respect to γ if $r_\gamma(\phi) < \infty$. Note that all Hamiltonian diffeomorphisms g are bounded since $r_\gamma(g) \leq 2\gamma(g) < \infty$. Similar to Proposition 1.2.A in [9], r_γ satisfies the triangle inequality $r_\gamma(\phi\psi) \leq r_\gamma(\phi) + r_\gamma(\psi)$. Thus two symplectomorphisms with the same flux are either both bounded or both unbounded.

From the above discussion, it is clear that to prove $\text{BI}_0(M, \omega, \gamma) = \text{Ham}(M, \omega)$, it suffices to show that for each nonzero value $v \in H^1(M, \mathbb{R}) / \Gamma_\omega$, there exists some unbounded element $\phi \in \text{Symp}_0(M, \omega)$ with $\text{Flux}(\phi) = v$.

A linear symplectic form ω_L on \mathbb{T}^{2n} is a 2-form $\omega_L = \sum_{i < j} a_{ij} dx_i \wedge dx_j$ such that ω_L^n never vanishes. Let $\{\phi_\theta^i\} \in \pi_1(\text{Symp}_0(\mathbb{T}^{2n}, \omega))$ be the loop of rotations of \mathbb{T}^{2n} along x_i direction. One easily computes

$$\xi_i := \text{Flux}(\{\phi_\theta^i\}) = \sum_{j=1}^{2n} a_{ij} dx_j.$$

Here we take the convention that $a_{ij} = -a_{ji}$. In particular, $a_{ii} = 0$. The following lemma can be proved exactly the same way as Lemma 7.2 in [6].

Lemma 4.1. *For \mathbb{T}^{2n} with the linear symplectic form $\omega_L := \sum_{i < j} a_{ij} dx_i \wedge dx_j$, the flux subgroup $\Gamma_{\omega_L} \subset H^1(\mathbb{T}^{2n}, \mathbb{R})$ is generated by the above ξ_i 's over \mathbb{Z} . That is, $\Gamma = \mathbb{Z}\langle \xi_1, \xi_2, \dots, \xi_{2n} \rangle$.*

Proof. The proof is the same as the proof of Lemma 7.2 in [6], and is therefore omitted. \square

Proof of Theorem 1.7. Let $\phi \in \text{Symp}_0(\mathbb{T}^{2n}, \omega_L)$ such that

$$\phi(x_1, x_2, \dots, x_{2n}) = (x_1 + \alpha_1, x_2 + \alpha_2, \dots, x_{2n} + \alpha_{2n})$$

where $\alpha_i \in \mathbb{R}/\mathbb{Z}$ for $1 \leq i \leq 2n$. Then

$$\text{Flux}(\phi) = \sum_{i=1}^{2n} \alpha_i \xi_i.$$

As explained above, it suffices to show that ϕ is unbounded with respect to the spectral norm γ when at least one $\alpha_i \in \mathbb{R}/\mathbb{Z}$ is nonzero. Assume $\alpha_1 \neq 0$ without loss of generality. Thus $\phi(U) \cap U = \emptyset$ where $U \subset \mathbb{T}^{2n}$ is defined by

$$U := \{(x_1, x_2, \dots, x_{2n}) \in \mathbb{T}^{2n} \mid |x_1| < \epsilon\}.$$

for sufficiently small ϵ .

As in the proof of Theorem 1.1, for every $c \in \mathbb{R}$, let F_c be a time-independent Hamiltonian function of \mathbb{T}^{2n} supported in U which has exactly two critical values, namely, 0 and its maximum value c . We also require that F_c depend only on the first coordinate x_1 . Denote by $f_c \in \text{Ham}(\mathbb{T}^{2n}, \omega_L)$ the time-1 map of the Hamiltonian F_c . Since $\phi(U) \cap U = \emptyset$, we know that

$$h_c := f_c \phi f_c^{-1} \phi^{-1} \in \text{Ham}(\mathbb{T}^{2n}, \omega_L)$$

is supported in the union of two disjoint sets $U \cup \phi(U)$, and is the time-1 map of the Hamiltonian $H_c = F_c - F_c \circ \phi^{-1}$, which has exactly three critical values $c, 0, -c$. Using the fact that ω_L is a linear symplectic form, one easily finds that the only contractible 1-periodic solutions of (2.1) are the critical points of H_c . Thus the spectral norm $\gamma(h_c)$ of h_c goes to infinity as c goes to infinity, which implies ϕ is unbounded with respect to γ . This proves Theorem 1.7. \square

Remark 4.2. One may attempt to apply Theorem 1.1 by showing that ϕ disjoins some Lagrangian torus $L \subset \mathbb{T}^{2n}$ from itself. For a general linear symplectic form ω_L , however, such Lagrangian tori may not even exist in \mathbb{T}^{2n} . We go around the difficulty by using the linearity of ω_L .

Proof of Theorem 1.4. We have shown in the proof of Theorem 1.7 above the existence of some unbounded symplectomorphism with respect to the spectral norm, which is sufficient to prove Theorem 1.4 as explained in Section 3. \square

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